ELSEVIER Applied Mathematics and Computation 102 (1999) 35–49

APPLIED MATHEMATICS AND COMPUTATION

# An elementary result in the stability theory of time-invariant nonlinear discrete dynamical systems

Weiye Li [a], Ferenc Szidarovszky [b,*]

[a] *Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA*
[b] *Systems and Industrial Engineering Department, University of Arizona, Tucson, AZ 85721, USA*

**Abstract**

The stability of the equilibria of time-invariant nonlinear dynamical systems with discrete time scale is investigated. We present an elementary proof showing that in the case of a stable equilibrium and continuously differentiable state transition function, all eigenvalues of the Jacobian computed at the equilibrium must be inside or on the unit circle. We also demonstrate via numerical examples that if some eigenvalues are on the unit circle and all other eigenvalues are inside the unit circle, then the equilibrium maybe unstable, or marginally stable, or even asymptotically stable, which show that the necessary condition cannot be further restricted in general. In addition, the necessary condition is given in terms of spectral radius and matrix norms. 

## 1. Introduction

Stationary dynamical systems with discrete time scale can be written as

$$\mathbf{x}_{k+1} = \mathbf{T}(\mathbf{x}_k), \tag{1}$$

where $\mathbf{x}_k$ is the state of the system at time period $k$, and $\mathbf{T}$ is the state-transition function. Let $X \subseteq \mathbb{R}^n$ and $\mathscr{R}(\mathbf{T}) \subseteq X$. For any initial state $\mathbf{x}_0 \in X$, Eq. (1) uniquely determines the state trajectory, $\mathbf{x}_k$, $k \geqslant 0$.

* Corresponding author. E-mail: szidar@sie.arizona.edu.


PII: S0096-3003(98)10013-9

A state $\bar{\mathbf{x}} \in X$ is called an *equilibrium* of system (1), if

$$\bar{\mathbf{x}} = \mathbf{T}(\bar{\mathbf{x}}), \tag{2}$$

that is, the equilibria of system (1) and the fixed points of mapping $\mathbf{T}$ are equivalent to each other.

An equilibrium $\bar{\mathbf{x}}$ is called *stable* or *marginally stable* if for arbitrary $\epsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}_0 - \bar{\mathbf{x}}\| < \delta$ implies that for all $k \geqslant 0$, $\|\mathbf{x}_k - \bar{\mathbf{x}}\| < \epsilon$. An equilibrium is *asymptotically stable* if it is marginally stable and there exists a $\Delta > 0$ such that $\|\mathbf{x}_0 - \bar{\mathbf{x}}\| < \Delta$ implies that $\mathbf{x}_k \to \bar{\mathbf{x}}$ as $k \to \infty$. An equilibrium is *globally asymptotically stable* if it is marginally stable and $\mathbf{x}_k \to \bar{\mathbf{x}}$ as $k \to \infty$ with arbitrary initial state $\mathbf{x}_0 \in X$.

There are many sufficient conditions for the marginal, asymptotical, or global asymptotical stability of an equilibrium. Criteria based on Lyapunov functions are discussed in almost all textbooks on dynamical systems theory (see, for example, [1]). Recently, Li et al. [2] have developed necessary stability conditions based on Lyapunov functions.

In this paper we will focus on criteria based on the spectral properties of the Jacobian of the state transition function. It is well known that if $\bar{\mathbf{x}}$ is an interior point and $\mathbf{T}$ is continuously differentiable in the neighbourhood of $\bar{\mathbf{x}}$ and all eigenvalues of $\mathbf{J}(\bar{\mathbf{x}})$ (which is the Jacobian at the equilibrium) are inside the unit circle, then $\bar{\mathbf{x}}$ is asymptotically stable. Necessary stability conditions have been obtained by using stable-unstable manifold theory [3], and some of the conditions can be stated in an elementary way. However, no elementary proofs of these results are offered in the literature. In this paper we present an elementary proof showing that if $\bar{\mathbf{x}}$ is marginally stable, then all eigenvalues of $\mathbf{J}(\bar{\mathbf{x}})$ must be on or inside the unit circle. Since asymptotical stability implies marginal stability, this necessary condition applies also to asymptotically stable equilibrium as well. Via numerical examples we will also demonstrate that in the case when some eigenvalues have unit absolute values and all other eigenvalues are inside the unit circle, then $\bar{\mathbf{x}}$ can be either unstable, or marginally stable, or even asymptotically stable showing that the necessary condition cannot be further restricted. In the last section of the paper the necessary condition will be derived in terms of spectral radius and matrix norms.

## 2. A necessary condition for stability

Before formulating the main theorem of this paper, some preliminary results are presented.

**Lemma 2.1** (A canonical form of a real matrix). *For $l = 1, 2, \ldots,$ denote*

$$\mathbf{N}_l = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}_{l \times l}, \qquad \mathbf{M}_l = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}_{l \times l},$$

*and*

$$\mathbf{L}_{2l} = \begin{pmatrix} (\cos\,\theta_l)\mathbf{M}_l & (\sin\,\theta_l)\mathbf{M}_l \\ -(\sin\,\theta_l)\mathbf{M}_l & (\cos\,\theta_l)\mathbf{M}_l \end{pmatrix}_{2l \times 2l}.$$

*For every real square matrix* $\mathbf{A}$, *there is a nonsingular real matrix* $\mathbf{P}$ *such that*

$$\begin{aligned} \Lambda &= \mathbf{PAP}^{-1} \\ &= \mathrm{diag}(\mathbf{N}_{e_1}, \ldots, \mathbf{N}_{e_r}, \lambda_{r+1}\mathbf{M}_{e_{r+1}}, \ldots, \lambda_{r+s}\mathbf{M}_{e_{r+s}}, |\tau_{r+s+1}|\mathbf{L}_{2e_{r+s+1}}, \ldots, |\tau_{r+s+t}|\mathbf{L}_{2e_{r+s+t}}), \end{aligned}$$

*where* $\mathbf{N}_{e_i}$ $(1 \leqslant i \leqslant r)$ *are the blocks corresponding to the zero eigenvalue,* $\lambda_i\mathbf{M}_{e_i}$ $(r+1 \leqslant i \leqslant r+s)$ *are the blocks corresponding to the real eigenvalues* $\lambda_i$, $|\tau_i|\mathbf{L}_{2e_i}$ $(r+s+1 \leqslant i \leqslant r+s+t)$ *are the blocks corresponding to the nonreal eigenvalues* $\tau_i$.

**Proof.** The following proof is based on Ref. [4].

Any matrix is similar to its Jordan form. If there are zero eigenvalues, then the Jordan blocks corresponding to them are $\mathbf{N}_{e_1}, \ldots, \mathbf{N}_{e_r}$. Any Jordan block corresponding to any nonzero eigenvalue $\lambda$ can be transformed as

$$\begin{pmatrix} 1 & & & & \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda^{l-2} & \\ & & & & \lambda^{l-1} \end{pmatrix}^{-1} \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \times \begin{pmatrix} 1 & & & & \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda^{l-2} & \\ & & & & \lambda^{l-1} \end{pmatrix} = \lambda\mathbf{M}_l.$$

Therefore all Jordan blocks corresponding to real nonzero eigenvalues are similar to $\lambda_{r+1}\mathbf{M}_{e_{r+1}}, \ldots, \lambda_{r+s}\mathbf{M}_{e_{r+s}}$.

The characteristic polynomial of any real matrix $\mathbf{A}$ is a real polynomial, so are its invariant factors. Then any nonreal elementary factor and its conjugate factor must appear in pairs. The above discussion implies that all Jordan blocks of $\mathbf{A}$ corresponding to nonreal eigenvalues must be similar to matrix

$$\mathrm{diag}(\tau_{r+s+1}\mathbf{M}_{2e_{r+s+1}}, \bar{\tau}_{r+s+1}\mathbf{M}_{2e_{r+s+1}}, \ldots, \tau_{r+s+t}\mathbf{M}_{2e_{r+s+t}}, \bar{\tau}_{r+s+t}\mathbf{M}_{2e_{r+s+t}}).$$

Notice that with the notation $\tau_l = |\tau_l|(\cos\,\theta_l + \mathrm{i}\,\sin\,\theta_l)$,

$$\begin{aligned}
&\begin{pmatrix} \mathbf{I}_l & -\mathrm{i}\mathbf{I}_l \\ \mathbf{I}_l & \mathrm{i}\mathbf{I}_l \end{pmatrix} (|\tau_l|\mathbf{L}_{2l}) \begin{pmatrix} \mathbf{I}_l & -\mathrm{i}\mathbf{I}_l \\ \mathbf{I}_l & \mathrm{i}\mathbf{I}_l \end{pmatrix}^{-1} \\
&\quad = |\tau_l| \begin{pmatrix} \mathbf{I}_l & -\mathrm{i}\mathbf{I}_l \\ \mathbf{I}_l & \mathrm{i}\mathbf{I}_l \end{pmatrix} \begin{pmatrix} (\cos\,\theta_l)\mathbf{M}_l & (\sin\,\theta_l)\mathbf{M}_l \\ \\ -(\sin\,\theta_l)\mathbf{M}_l & (\cos\,\theta_l)\mathbf{M}_l \end{pmatrix} \begin{pmatrix} \mathbf{I}_l & -\mathrm{i}\mathbf{I}_l \\ \mathbf{I}_l & \mathrm{i}\mathbf{I}_l \end{pmatrix}^{-1} \\
&\quad = \begin{pmatrix} \tau_l\mathbf{M}_l & \\ & \bar{\tau}_l\mathbf{M}_l \end{pmatrix}.
\end{aligned}$$

Since $\mathbf{A}$ and $\lambda$ are real, they are real similar. □

**Lemma 2.2.** *Suppose $\{\mathbf{x}_k : k = 0, 1, 2, \ldots\}$ is the trajectory of a discrete dynamic system obtained by $\mathbf{x}_{k+1} = \mathbf{T}(\mathbf{x}_k)$ in $X \subseteq \mathbb{R}^n$. Let $\mathbf{P}$ be a nonsingular real matrix. Then under the linear transformation $\Pi$: $\mathbf{y} \overset{\mathrm{def}}{=} \Pi(\mathbf{x}) = \mathbf{P}\mathbf{x}$, the image of the original dynamical system is a discrete dynamical system in $Y \overset{\mathrm{def}}{=} \Pi(X)$ obtained by $\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{y}_k)$, where $\mathbf{h} = \Pi \circ \mathbf{T} \circ \Pi^{-1}$. In particular, if $\mathbf{T} \in C^1(X)$, then $\mathbf{h} \in C^1(Y)$.*

**Proof.** The assertion follows from relation

$$\mathbf{y}_{k+1} = \mathbf{P}\mathbf{x}_{k+1} = \mathbf{P}\mathbf{T}(\mathbf{x}_k) = \mathbf{P}\mathbf{T}(\mathbf{P}^{-1}\mathbf{y}_k) = (\Pi \circ \mathbf{T} \circ \Pi^{-1})(\mathbf{y}_k),$$

for $k = 0, 1, 2, \ldots$ In particular, if $\mathbf{T} \in C^1(X)$, then using the chain rule of differentiation, we see that $\mathbf{h} \in C^1(Y)$. □

**Lemma 2.3.** *Stability (or asymptotical stability) of an equilibrium is preserved under linear transformation. Namely, $\bar{\mathbf{x}}$ is a stable (or asymptotically stable) equilibrium of a dynamic system $\mathbf{x}_{k+1} = \mathbf{T}(\mathbf{x}_k)$ in $X \subseteq \mathbb{R}^n$ if and only if under the linear transformation $\Pi$: $\mathbf{y} \overset{\mathrm{def}}{=} \Pi(\mathbf{x}) = \mathbf{P}\mathbf{x}$ where $\mathbf{P}$ is a nonsingular real matrix, $\bar{\mathbf{y}} \overset{\mathrm{def}}{=} \mathbf{P}\bar{\mathbf{x}}$ is a stable (or asymptotically stable) equilibrium of the dynamical system $\mathbf{y}_{k+1} = (\Pi \circ \mathbf{T} \circ \Pi^{-1})(\mathbf{y}_k)$.*

**Proof.** Notice that

$$\|\mathbf{y}_k - \bar{\mathbf{y}}\| = \|\mathbf{P}\mathbf{x}_k - \mathbf{P}\bar{\mathbf{x}}\| = \|\mathbf{P}(\mathbf{x}_k - \bar{\mathbf{x}})\| \leqslant \|\mathbf{P}\|\,\|\mathbf{x}_k - \bar{\mathbf{x}}\|,$$

and

$$\|\mathbf{x}_k - \bar{\mathbf{x}}\| = \|\mathbf{P}^{-1}\mathbf{y}_k - \mathbf{P}^{-1}\bar{\mathbf{y}}\| = \|\mathbf{P}^{-1}(\mathbf{y}_k - \bar{\mathbf{y}})\| \leqslant \|\mathbf{P}^{-1}\|\,\|\mathbf{y}_k - \bar{\mathbf{y}}\|,$$

from which the assertion follows immediately. □

**Lemma 2.4.** *Let $\mathbf{N}_l$, $\mathbf{M}_l$ and $\mathbf{L}_{2l}$ be defined as in Lemma 2.1. Then for $k = 1, 2, \ldots,$*

$$\mathbf{N}_l^k = \begin{cases} \text{Matrix with the only nonzero entries } 1 \text{ on the} \\ \quad \text{upper } (k+1)\text{th subdiagonal} & \text{if } k < l, \\ \mathbf{O} \ (\text{zero matrix}) & \text{if } k \geqslant l; \end{cases}$$

$$\mathbf{M}_l^k = \begin{bmatrix} C_k^0 & C_k^1 & C_k^2 & \cdots & C_k^{l-1} \\ & C_k^0 & C_k^1 & \cdots & C_k^{l-2} \\ & & \ddots & \ddots & \vdots \\ & & & C_k^0 & C_k^1 \\ & & & & C_k^0 \end{bmatrix}_{l\times l},$$

$$\mathbf{M}_l^{-k} = \begin{bmatrix} C_{k-1}^0 & -C_k^1 & C_{k+1}^2 & \cdots & (-1)^{l-1}C_{k+l-2}^{l-1} \\ & C_{k-1}^0 & -C_k^1 & \cdots & (-1)^{l-2}C_{k+l-3}^{l-2} \\ & & \ddots & \ddots & \vdots \\ & & & C_{k-1}^0 & -C_k^1 \\ & & & & C_{k-1}^0 \end{bmatrix}_{l\times l},$$

*where $C_k^i$ is the binomial number $\binom{k}{i}$;*

$$\mathbf{L}_{2l}^k = \begin{bmatrix} (\cos k\theta_l)\mathbf{M}_l^k & (\sin k\theta_l)\mathbf{M}_l^k \\ -(\sin k\theta_l)\mathbf{M}_l^k & (\cos k\theta_l)\mathbf{M}_l^k \end{bmatrix}_{2l\times 2l},$$

$$\mathbf{L}_{2l}^{-k} = \begin{bmatrix} (\cos k\theta_l)\mathbf{M}_l^{-k} & -(\sin k\theta_l)\mathbf{M}_l^{-k} \\ (\sin k\theta_l)\mathbf{M}_l^{-k} & (\cos k\theta_l)\mathbf{M}_l^{-k} \end{bmatrix}_{2l\times 2l}.$$

**Proof.** Easy calculation and finite induction can be applied to show these relations. For example, in showing the closed form representation for $\mathbf{M}_l^{-k}$ using induction we need the identity

$$C_{k-1}^0 + C_k^1 + \cdots + C_{k+l-1}^l = C_{k+l}^l,$$

which follows immediately from the repeated application of the additive property of the binomial numbers, since

$$C_{k-1}^0 + C_k^1 + \cdots + C_{k+l-1}^l = (C_k^0 + C_k^1) + \cdots + C_{k+l-1}^l$$
$$= (C_{k+1}^1 + C_{k+1}^2) + \cdots + C_{k+l-1}^l = \cdots = C_{k+l-1}^{l-1} + C_{k+l-1}^l = C_{k+l}^l. \qquad \square$$

**Corollary 2.5.** *Let $\|.\|_1$ denote the column-norm of real or complex matrices. Then for $k = 1, 2, \ldots$,*

$$\|\mathbf{N}_l^k\|_1 \leqslant 1 < lk^{l-1}, \qquad \|\mathbf{M}_l^k\|_1 \leqslant lk^{l-1}, \qquad \|\mathbf{M}_l^{-k}\|_1 \leqslant l(k+l-2)^{l-1},$$

$$\|\mathbf{L}_{2l}^k\|_1 \leqslant (|\cos k\theta_l| + |\sin k\theta_l|)\|\mathbf{M}_l^k\|_1 \leqslant \sqrt{2}\|\mathbf{M}_l^k\|_1 \leqslant \sqrt{2}lk^{l-1} < 2lk^{2l-1},$$

$$\|\mathbf{L}_{2l}^{-k}\|_1 \leqslant (|\cos k\theta_l| + |\sin k\theta_l|)\|\mathbf{M}_l^{-k}\|_1 \leqslant \sqrt{2}\|\mathbf{M}_l^{-k}\|_1$$
$$\leqslant \sqrt{2}l(k+l-2)^{l-1} < 2l(k+2l-2)^{2l-1}.$$

We are now able to formulate the main result of this paper.

**Theorem 2.6.** *Let $X$ be a convex open set in $\mathbb{R}^n$. Assume that for mapping $\mathbf{T} \in C^1(X)$, $\mathscr{R}(\mathbf{T}) \subseteq X$, and the discrete dynamical system $\mathbf{x}_{k+1} = \mathbf{T}(\mathbf{x}_k)$ has an equilibrium $\bar{\mathbf{x}} \in X$. Then the necessary condition for the stability of $\bar{\mathbf{x}}$ is that the absolute values of the eigenvalues of $\mathbf{J} = (\partial\mathbf{T}/\partial\mathbf{x})|_{\mathbf{x}=\bar{\mathbf{x}}}$, the Jacobian of* $\mathbf{T}$ at $\bar{\mathbf{x}}$, *are less than or equal to one.*

**Proof.** Since translations do not change the Jacobian of $\mathbf{T}$ and preserve the stability of an equilibrium, we may assume that $X$ contains $\mathbf{0}$ and $\bar{\mathbf{x}} = \mathbf{0}$. If $\bar{\mathbf{x}} \neq \mathbf{0}$, then introduce the new state variable $\mathbf{x}' = \mathbf{x} - \bar{\mathbf{x}}$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{J}$. Let $\alpha = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$ and assume that $\alpha > 1$. Define

$$\beta = \begin{cases} -\infty, & \text{if } |\lambda_1| = |\lambda_2| = \cdots = |\lambda_n| = \alpha; \\ \max\{\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\} \setminus \{\alpha\}\}, & \text{otherwise.} \end{cases}$$

By Lemma 2.1, there exists a nonsingular real matrix $\mathbf{P}$ such that $\mathbf{PJP}^{-1} = \Lambda$, where $\Lambda$ is of the form as given in Lemma 2.1. Since interchanging the same rows and columns does not affect the similarity of matrices, we can write $\tilde{\Lambda}$ in the following form obtained by rearranging the orders of the diagonal blocks of $\Lambda$:

$$\tilde{\Lambda} = \mathbf{QJQ}^{-1} = \mathrm{diag}\{\mu_1\mathbf{B}_{f_1}, \mu_2\mathbf{B}_{f_2}, \ldots, \mu_p\mathbf{B}_{f_p}, \ldots, \mu_q\mathbf{B}_{f_q}\},$$

where $\mathbf{Q}$ is real and nonsingular; $1 \leqslant p \leqslant q$; for all $i$ ($1 \leqslant i \leqslant q$), $\mathbf{B}_{f_i}$ is an $f_i \times f_i$ matrix in form of either $\mathbf{N}_{f_i}$, $\mathbf{M}_{f_i}$ or $\mathbf{L}_{f_i}$ (in the last case, $f_i$ is even); $|\mu_i| = \alpha$ for $1 \leqslant i \leqslant p$, $|\mu_i| \leqslant \beta$ for $p+1 \leqslant i \leqslant q$. (For blocks corresponding to zero eigenvalues we can select $\mu_j = 1$ and $\mathbf{B}_{f_j} = \mathbf{N}_{f_j}$.) Obviously, $p = q$ corresponds to the case when $\beta = -\infty$.

Introduce the notation $\tilde{\mathbf{B}}_{f_i} = \mathrm{diag}\{\mathbf{O}, \ldots, \mathbf{O}, \mathbf{B}_{f_i}, \mathbf{O}, \ldots, \mathbf{O}\}$ for $1 \leqslant i \leqslant q$.

We will next define a norm in $\mathbb{R}^n$. For any $\mathbf{z} \in \mathbb{R}^n$, we can uniquely write

$$\mathbf{z} = (\mathbf{z})_1 + (\mathbf{z})_2 + \cdots + (\mathbf{z})_q,$$

where $(\mathbf{z})_i$ is the component vector of $\mathbf{z}$ associated to the block $\mathbf{B}_{f_i}$ in $\tilde{\mathbf{B}}_{f_i}$, $1 \leqslant i \leqslant q$. Therefore $(\mathbf{z})_i$ is an $f_i$-dimensional vector. Let $\|.\|_{1,j}$ denote the $l^1$-norm in $\mathbb{R}^j$, $1 \leqslant j < \infty$. Define $\|.\|$ as follows:

$$\|\mathbf{z}\| = \|(\mathbf{z})_1\|_{1,f_1} + \|(\mathbf{z})_2\|_{1,f_2} + \cdots + \|(\mathbf{z})_q\|_{1,f_q}.$$

Clearly, $\|.\|$ well defines a norm, and it coincides with the $l^1$-norm in $\mathbb{R}^n$. Note that the column-norm of a matrix is compliant to the $l^1$-norm of a vector space.

Let $f = \max\{f_1, f_2, \ldots, f_q\}$. Recall that for any positive $a$, $b$, $c$ such that $a > \max\{1, b\}$, we have $\lim_{k\to\infty}(a^k/b^k) = +\infty$ and $\lim_{k\to\infty}(a^k/k^c) = +\infty$. Notice that $\alpha > \max\{1, \beta\}$. Therefore, there is a positive integer $k_0$, such that

$$\frac{\alpha^{k_0}}{f(k_0 + f - 2)^{f-1}} - 2 > \max\{4,\ \beta^{k_0}\ f\ k_0^{f-1} + 2\}.$$

(This is also true when $\beta = -\infty$ with an odd value of $k_0$.)

Let $\mathbf{S} = \mathbf{T}^{k_0}$, then the state trajectory of the dynamical system $\mathbf{z}_{k+1} = \mathbf{S}(\mathbf{z}_k)$ with $\mathbf{z}_0 = \mathbf{x}_0$ in $X$ is a subsequence of the state sequence of the original system $\mathbf{x}_{k+1} = \mathbf{T}(\mathbf{x}_k)$. By the chain rule of differentiation,

$$\frac{\partial \mathbf{S}}{\partial \mathbf{x}}(\mathbf{x}) = \frac{\partial \mathbf{T}^{k_0}}{\partial \mathbf{x}}(\mathbf{x}) = \frac{\partial \mathbf{T}}{\partial \mathbf{x}}(\mathbf{T}^{k_0-1}(\mathbf{x}))\ \frac{\partial \mathbf{T}}{\partial \mathbf{x}}(\mathbf{T}^{k_0-2}(\mathbf{x}))\cdots\frac{\partial \mathbf{T}}{\partial \mathbf{x}}(\mathbf{T}(\mathbf{x}))\ \frac{\partial \mathbf{T}}{\partial \mathbf{x}}(\mathbf{x}).$$

Note that $\bar{\mathbf{x}} = \mathbf{0}$ is a fixed point of $\mathbf{T}$, therefore

$$\frac{\partial \mathbf{S}}{\partial \mathbf{x}}(\mathbf{0}) = \left[\frac{\partial \mathbf{T}}{\partial \mathbf{x}}(\mathbf{0})\right]^{k_0} = \mathbf{J}^{k_0}.$$

Now we perform the linear transformation $\Pi$: $X \mapsto Y \stackrel{\text{def}}{=} \Pi(X)$, by $\mathbf{y} \stackrel{\text{def}}{=} \Pi(\mathbf{x}) = \mathbf{Q}\mathbf{x}$. By Lemmas 2.2 and 2.3, the dynamic system $\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{y}_k)$ has a stable equilibrium $\bar{\mathbf{y}} = \mathbf{0}$, where $\mathbf{h} = \Pi \circ \mathbf{S} \circ \Pi^{-1}$ is continuously differentiable. The Jacobian of $\mathbf{h}$ at $\mathbf{0}$ is

$$\frac{\partial \mathbf{h}}{\partial \mathbf{y}}(\mathbf{0}) = \mathbf{Q}\ \frac{\partial \mathbf{S}}{\partial \mathbf{x}}(\mathbf{0})\ \mathbf{Q}^{-1} = \mathbf{Q}\mathbf{J}^{k_0}\mathbf{Q}^{-1} = \tilde{\Lambda}^{k_0}.$$

By the local linearization of $\mathbf{h}$, we have

$$\mathbf{h}(\mathbf{y}) = \mathbf{h}(\mathbf{0}) + \frac{\partial \mathbf{h}}{\partial \mathbf{y}}(\mathbf{0})\ (\mathbf{y} - \mathbf{0}) + \mathrm{o}(\|\mathbf{y}\|) = \tilde{\Lambda}^{k_0}\mathbf{y} + \mathrm{o}(\|\mathbf{y}\|).$$

Hence there exists a $\delta > 0$ such that whenever $\|\mathbf{y}\| < \delta$,

$$\|\mathbf{h}(\mathbf{y}) - \tilde{\Lambda}^{k_0}\mathbf{y}\| < \|\mathbf{y}\|. \tag{3}$$

Define

$$B = \{\mathbf{y}\colon\ \|\mathbf{y}\| < \delta\},$$

and

$$C = \begin{cases} \{\mathbf{y}\colon \|(\mathbf{y})_1\|_{1,f_1} + \cdots + \|(\mathbf{y})_p\|_{1,f_p} \\ \quad > \|(\mathbf{y})_{p+1}\|_{1,f_{p+1}} + \cdots + \|(\mathbf{y})_q\|_{1,f_q}\} & \text{if } p < q, \\ Y & \text{if } p = q. \end{cases}$$

Note that $C$ can be rewritten as

$$C = \begin{cases} \{\mathbf{y}\colon \|(\mathbf{y})_1\|_{1,f_1} + \cdots + \|(\mathbf{y})_p\|_{1,f_p} \\ \quad > \frac{1}{2}\|\mathbf{y}\| > \|(\mathbf{y})_{p+1}\|_{1,f_{p+1}} + \cdots + \|(\mathbf{y})_q\|_{1,f_q}\} & \text{if } p < q, \\ Y & \text{if } p = q. \end{cases}$$

From now on, we only consider the case when $p < q$ since the case of $p = q$ can be treated similarly.

Select any $\mathbf{y}^* \in B \cap C$. From inequality (3) and the construction of $\|.\|$ we have

$$\sum_{i=1}^{q} \|(\mathbf{h}(\mathbf{y}^*))_i - (\tilde{\Lambda}^{k_0}\mathbf{y}^*)_i\|_{1,f_i} < 2\sum_{i=1}^{p} \|(\mathbf{y}^*)_i\|_{1,f_i}.$$

So

$$\sum_{i=1}^{p} \|(\mathbf{h}(\mathbf{y}^*))_i - (\tilde{\Lambda}^{k_0}\mathbf{y}^*)_i\|_{1,f_i} < 2\sum_{i=1}^{p} \|(\mathbf{y}^*)_i\|_{1,f_i},$$

and

$$\sum_{i=p+1}^{q} \|(\mathbf{h}(\mathbf{y}^*))_i - (\tilde{\Lambda}^{k_0}\mathbf{y}^*)_i\|_{1,f_i} < 2\sum_{i=1}^{p} \|(\mathbf{y}^*)_i\|_{1,f_i}.$$

Let $f = \max\{f_1, f_2, \ldots, f_q\}$. By the triangle inequality and the corollary of Lemma 2.4 we have

$$\begin{aligned} \sum_{i=1}^{p} \|(\mathbf{h}(\mathbf{y}^*))_i\|_{1,f_i} &> \sum_{i=1}^{p} \|(\tilde{\Lambda}^{k_0}\mathbf{y}^*)_i\|_{1,f_i} - 2\sum_{i=1}^{p} \|(\mathbf{y}^*)_i\|_{1,f_i} \\ &= \sum_{i=1}^{p} \|\mu_i^{k_0}\tilde{\mathbf{B}}_{f_i}^{k_0}(\mathbf{y}^*)_i\|_{1,f_i} - 2\sum_{i=1}^{p} \|(\mathbf{y}^*)_i\|_{1,f_i} \\ &= \alpha^{k_0}\sum_{i=1}^{p} \|\tilde{\mathbf{B}}_{f_i}^{k_0}(\mathbf{y}^*)_i\|_{1,f_i} - 2\sum_{i=1}^{p} \|(\mathbf{y}^*)_i\|_{1,f_i} \\ &\geqslant \alpha^{k_0}\sum_{i=1}^{p} \frac{\|(\mathbf{y}^*)_i\|_{1,f_i}}{\|\tilde{\mathbf{B}}_{f_i}^{-k_0}\|_1} - 2\sum_{i=1}^{p} \|(\mathbf{y}^*)_i\|_{1,f_i} \\ &\geqslant \alpha^{k_0}\sum_{i=1}^{p} \frac{\|(\mathbf{y}^*)_i\|_{1,f_i}}{f_i(k_0 + f_i - 2)^{f_i - 1}} - 2\sum_{i=1}^{p} \|(\mathbf{y}^*)_i\|_{1,f_i} \\ &\geqslant \left[\frac{\alpha^{k_0}}{f(k_0 + f - 2)^{f-1}} - 2\right] \sum_{i=1}^{p} \|(\mathbf{y}^*)_i\|_{1,f_i}\ , \end{aligned}$$

and by using the corollary of Lemma 2.4 again we see that

$$\begin{aligned}
\sum_{i=p+1}^{q}\|(\mathbf{h}(\mathbf{y}^*))_i\|_{1,f_i} &< \sum_{i=p+1}^{q}\|(\tilde{\Lambda}^{k_0}\mathbf{y}^*)_i\|_{1,f_i} + 2\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} \\
&= \sum_{i=p+1}^{q}\|\mu_i^{k_0}\tilde{\mathbf{B}}_{f_i}^{k_0}(\mathbf{y}^*)_i\|_{1,f_i} + 2\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} \\
&\leqslant \beta^{k_0}\sum_{i=p+1}^{q}\|\tilde{\mathbf{B}}_{f_i}^{k_0}(\mathbf{y}^*)_i\|_{1,f_i} + 2\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} \\
&\leqslant \beta^{k_0}\sum_{i=p+1}^{q}\|\tilde{\mathbf{B}}_{f_i}^{k_0}\|_1\|(\mathbf{y}^*)_i\|_{1,f_i} + 2\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} \\
&\leqslant \beta^{k_0}\sum_{i=p+1}^{q} f_i k_0^{f_i-1}\|(\mathbf{y}^*)_i\|_{1,f_i} + 2\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} \\
&\leqslant \beta^{k_0} f k_0^{f-1}\sum_{i=p+1}^{q}\|(\mathbf{y}^*)_i\|_{1,f_i} + 2\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} \\
&\leqslant \beta^{k_0} f k_0^{f-1}\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} + 2\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} \\
&= (\beta^{k_0} f k_0^{f-1} + 2)\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} .
\end{aligned}$$

By the choice of $k_0$, we have

$$\sum_{i=1}^{p}\|(\mathbf{h}(\mathbf{y}^*))_i\|_{1,f_i} > \sum_{i=p+1}^{q}\|(\mathbf{h}(\mathbf{y}^*))_i\|_{1,f_i} ,$$

and furthermore,

$$\begin{aligned}
\|\mathbf{h}(\mathbf{y}^*)\| &> \sum_{i=1}^{p}\|(\mathbf{h}(\mathbf{y}^*))_i\|_{1,f_i} > \left[\frac{\alpha^{k_0}}{f(k_0+f-2)^{f-1}} - 2\right]\sum_{i=1}^{p}\|(\mathbf{y}^*)_i\|_{1,f_i} \\
&> \frac{1}{2}\left[\frac{\alpha^{k_0}}{f(k_0+f-2)^{f-1}} - 2\right]\|\mathbf{y}^*\| > 2\|\mathbf{y}^*\| . \qquad (4)
\end{aligned}$$

Thus, $\mathbf{h}(\mathbf{y}^*) \in C$, and $\mathbf{h}(\mathbf{y}^*)$ is away from the equilibrium at least twice as much as $\mathbf{y}^*$.

Assume that the zero equilibrium is stable. Then there is a positive $\epsilon \leqslant \delta$ such that if $\|\mathbf{y}_0\| < \epsilon$ then for all $k \geqslant 0$, $\|\mathbf{y}_k\| < \delta$. If one selects a nonzero $\mathbf{y}_0 \in C$ such that $\|\mathbf{y}_0\| < \epsilon$, then inequality (3) holds for all $\mathbf{y}_k$ and obviously, $\mathbf{y}_k \in B \cap C$. However, from relation (4) we conclude that

$$\|\mathbf{y}_k\| > 2\,\|\mathbf{y}_{k-1}\| > \cdots > 2^k\,\|\mathbf{y}_0\|,$$

which tends to $\infty$ as $k \to \infty$. This contradicts the selection of $\epsilon$. Hence the proof is completed. □

**Corollary 2.7.** *Since asymptotical stability implies marginal stability, Theorem 2.6 holds also for any asymptotically stable equilibrium.*

We know that if all eigenvalues of $\mathbf{J}$ are inside the unit circle, then the equilibrium is asymptotically stable. If some eigenvalues are on the unit circle and all others are inside the unit circle, then the equilibrium may be unstable and in some cases may be marginally stable, or even asymptotically stable as it is illustrated in the following examples.

**Example 2.8.** Consider the linear system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$

with $\mathbf{x}_k = (x_{k1}, x_{k2})^{\mathrm{T}}$ and

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that for $k \geqslant 0$,

$$\mathbf{A}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

and therefore

$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 = \begin{pmatrix} x_{01} + kx_{02} \\ x_{02} \end{pmatrix}.$$

If $x_{02} > 0$, then $x_{k1} \to \infty$. Hence the zero equilibrium is unstable.

**Example 2.9.** Consider the single dimensional linear system

$$x_{k+1} = -x_k$$

with a unique equilibrium $\bar{x} = 0$. Since for $k \geqslant 0$, $x_k = (-1)^k x_0$, the equilibrium is stable and the stability is not asymptotical.

**Example 2.10.** Consider now the single dimensional nonlinear system

$$x_{k+1} = g(x_k)$$

with

$$g(x) = x\mathrm{e}^{-x^2},$$

which has a unique equilibrium $\bar{x} = 0$. The Jacobian of $g$ at $\bar{x}$ has unique eigenvalue 1. Since $|x_k|$ is decreasing in $k$ and bounded by zero, $a = \lim_{k\to\infty} |x_k|$ exists and is finite. Moreover, it satisfies the equation $a = a\mathrm{e}^{-a^2}$, which implies $a = 0$. By the monotonicity and convergence of $|x_k|$ we conclude that the zero equilibrium is asymptotically stable.

## 3. Stability and matrix norms

In this section, we will apply the spectral theory in finite dimensional spaces to reveal some interesting connections between stability and matrix norms.

Let $\mathbf{A}$ be a real or complex $n \times n$ matrix. Let the eigenvalues of $\mathbf{A}$ be denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. The spectral radius of matrix $\mathbf{A}$ is defined as

$$r_\sigma(\mathbf{A}) = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}.$$

It is well known that for all induced matrix norms,

$$r_\sigma(\mathbf{A}) \leqslant \|\mathbf{A}\|$$

(see for example, Ref. [5] or [6].)

There are many special spectral properties of real or complex matrices known from the literature. The following lemmas are given in Ortega and Rheinboldt [7].

**Lemma 3.1.** *Let* $\|.\|$ *be an arbitrary norm on* $\mathbb{R}^n$ *(or* $\mathbb{C}^n$*) and* $\mathbf{P}$ *an arbitrary nonsingular,* $n \times n$, *real (or complex) matrix. Then the mapping defined by* $\mathbf{x} \mapsto \|\mathbf{x}\|' = \|\mathbf{P}\mathbf{x}\|$, *for all* $\mathbf{x} \in \mathbb{R}^n$ *(or* $\mathbb{C}^n$*), is a norm on* $\mathbb{R}^n$ (or $\mathbb{C}^n$). *Moreover, if* $\mathbf{A}$ *is a real (or complex)* $n \times n$ *matrix, then the induced matrix norm is given as*

$$\|\mathbf{A}\|' = \|\mathbf{P}\mathbf{A}\mathbf{P}^{-1}\|.$$

**Proof.** It is simple to verify that $\|.\|'$ is a norm. The second part of the lemma follows from equality

$$\|\mathbf{A}\|' = \sup_{\|\mathbf{x}\|'=1} \|\mathbf{A}\mathbf{x}\|' = \sup_{\|\mathbf{P}\mathbf{x}\|=1} \|\mathbf{P}\mathbf{A}\mathbf{x}\| = \sup_{\|\mathbf{y}\|=1} \|\mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{y}\| = \|\mathbf{P}\mathbf{A}\mathbf{P}^{-1}\|. \quad \square$$

**Lemma 3.2.** *Let* $\Lambda$ *be the Jordan form of an* $n \times n$ *matrix* $\mathbf{A}$. *Then for arbitrary* $\epsilon > 0$, $\mathbf{A}$ *is similar to a matrix* $\hat{\Lambda}$ *which is identical to* $\Lambda$ *except that each off-diagonal* 1 *is replaced by* $\epsilon$.

**Proof.** Without loss of generality we may assume that $\Lambda$ is an $n \times n$ Jordan block. Let $\mathbf{D}$ be the diagonal matrix $\mathrm{diag}(1, \epsilon, \ldots, \epsilon^{n-1})$, then $\mathbf{D}^{-1}\Lambda\mathbf{D} = \hat{\Lambda}$. Hence, $\mathbf{A}$ is similar to $\hat{\Lambda}$. $\square$

**Lemma 3.3.** *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Then for arbitrary* $\epsilon > 0$, *there is a norm on* $\mathbb{C}^n$ *such that for the induced matrix norm,*

$$\|\mathbf{A}\| \leqslant r_\sigma(\mathbf{A}) + \epsilon.$$

**Proof.** Let $\hat{\Lambda}$ be the modified Jordan form of $\mathbf{A}$ as given in Lemma 3.2. Then the column-norm of $\hat{\Lambda}$ satisfies relation $\|\hat{\Lambda}\|_1 \leqslant r_\sigma(\mathbf{A}) + \epsilon$, and using Lemma 3.1, the result follows. $\square$

The main result of this section can be formulated as follows.

**Theorem 3.4.** *Let* $\mathbf{A}$ *be an* $n \times n$ *real or complex matrix. Then*

$$r_\sigma(\mathbf{A}) = \inf\{\|\mathbf{A}\|\colon\ \|.\| \text{ is any induced matrix norm on } \mathbb{C}^n\}.$$

*Furthermore, if all the Jordan blocks of* $\mathbf{A}$ *corresponding to the eigenvalues with largest absolute value have size* $1 \times 1$, *then* $r_\sigma(\mathbf{A}) = \|\mathbf{A}\|$ *with some matrix norm; otherwise,* $r_\sigma(\mathbf{A}) < \|\mathbf{A}\|$ *for all matrix norms.*

**Proof.** The first part of the theorem follows immediately from Lemma 3.3.

Suppose that all the Jordan blocks corresponding to the eigenvalues of $\mathbf{A}$ with largest absolute value have size $1 \times 1$. If $\mathbf{A}$ is diagonable, then take the column-norm of the diagonal Jordan form of $\mathbf{A}$. Then it equals $r_\sigma(\mathbf{A})$, and is a norm of $\mathbf{A}$ by Lemma 3.1. If $\mathbf{A}$ is not diagonable, then there must be some other eigenvalues of $\mathbf{A}$ with Jordan blocks with sizes more than $1 \times 1$. Let $\beta$ be the maximum of the absolute values of all other eigenvalues of $\mathbf{A}$. Then $r_\sigma(\mathbf{A}) > \beta$. Take $\epsilon = r_\sigma(\mathbf{A}) - \beta$, and apply Lemma 3.2 to each Jordan block of $\mathbf{A}$ with this $\epsilon$ to obtain the modified Jordan form $\hat{\Lambda}$ for $\mathbf{A}$. Then $\|\hat{\Lambda}\|_1 = r_\sigma(\mathbf{A})$, which is a norm of $\mathbf{A}$ by Lemma 3.1.

Suppose that there is an $m \times m$ ($m \geqslant 2$) Jordan block of $\mathbf{A}$ corresponding to a dominant eigenvalue $\lambda$. If $\lambda = 0$, then the assertion is obvious, since for any $\mathbf{A} \neq \mathbf{O}$, $\|\mathbf{A}\| > 0$ for all norms. Assume next that $\lambda \neq 0$. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ be the natural basis of the coordinate space corresponding to this Jordan block. Then $\mathbf{A}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = (\lambda x_1 + x_2)\mathbf{e}_1 + \lambda x_2\mathbf{e}_2$. Assume that there is some norm $\|.\|$ such that $\|\mathbf{A}\| = \sup_{\|\mathbf{x}\| \neq 0}(\|\mathbf{A}\mathbf{x}\|/\|\mathbf{x}\|) = r_\sigma(\mathbf{A})$. Then for all vectors $\mathbf{x}$, $\|\mathbf{A}\mathbf{x}\| \leqslant r_\sigma(\mathbf{A})\|\mathbf{x}\|$. We will show that this is impossible. Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, then

$$\|(\lambda x_1 + x_2)\mathbf{e}_1 + \lambda x_2\mathbf{e}_2\| \leqslant r_\sigma(\mathbf{A})\|x_1\mathbf{e}_1 + x_2\mathbf{e}_2\|$$

for all complex $x_1$ and $x_2$. Let $x_1 = m$ ($m = 1, 2, \ldots$) and $x_2 = \lambda$, then we see that

$$\|(m+1)\lambda\mathbf{e}_1 + \lambda^2\mathbf{e}_2\| \leqslant r_\sigma(\mathbf{A})\|m\mathbf{e}_1 + \lambda\mathbf{e}_2\|.$$

Since $r_\sigma(\mathbf{A}) = |\lambda| > 0$,

$$\|(m+1)\mathbf{e}_1 + \lambda\mathbf{e}_2\| \leqslant \|m\mathbf{e}_1 + \lambda\mathbf{e}_2\|.$$

This relation implies that for all $M \geqslant 2$,

$$\|M\mathbf{e}_1 + \lambda\mathbf{e}_2\| \leqslant \|(M-1)\mathbf{e}_1 + \lambda\mathbf{e}_2\| \leqslant \cdots \leqslant \|\mathbf{e}_1 + \lambda\mathbf{e}_2\|,$$

and division by $M$ yields

$$\left\|\mathbf{e}_1 + \frac{\lambda}{M}\mathbf{e}_2\right\| \leqslant \frac{1}{M}\|\mathbf{e}_1 + \lambda\mathbf{e}_2\|.$$

Letting $M \to \infty$ and using the continuity of vector norms we have $\|\mathbf{e}_1\| \leqslant 0$, which is impossible since with any vector norm, $\|\mathbf{e}_1\| > 0$.

Thus, the proof is completed. $\square$

From Theorems 2.6 and 3.4, we have the following interesting result.

**Corollary 3.5.** *Assume that all conditions of Theorem 2.6 hold. Then for a stable equilibrium, the spectual radius of* **J**, *the Jacobian of the transition function at the equilibrium, must not be greater than 1. Furthermore, if all the Jordan blocks of* **J** *corresponding to the eigenvalues with largest absolute value have size* $1 \times 1$, *then there exists some matrix norm such that* $\|\mathbf{J}\| \leqslant 1$.

The result of Corollary 3.5 cannot be further extended, as it is illustrated in the following example which provides a nonlinear discrete dynamic system with an asymptotically stable equilibrium and with all matrix norms of the Jacobian of the transition function at the equilibrium being strictly greater than one.

**Example 3.6.** Consider the dynamical system $\mathbf{z}_{t+1} = \mathbf{T}(\mathbf{z}_t)$, where $\mathbf{z} = (x, y)^{\mathrm{T}} \in \mathbb{R}^2$ and

$$\mathbf{T}(\mathbf{z}) = \begin{pmatrix} x\mathrm{e}^{-x^2} + y\mathrm{e}^{-y^2} \\ y\mathrm{e}^{-y^2} \end{pmatrix}. \tag{5}$$

The equilibrium is the solution of equations

$$x = x\mathrm{e}^{-x^2} + y\mathrm{e}^{-y^2}, \qquad y = y\mathrm{e}^{-y^2}.$$

From the second equation we see $y = 0$, and then the first equation implies that $x = 0$. Thus, the unique equilibrium is $\bar{\mathbf{z}} = \mathbf{0}$.

Notice that

$$\mathbf{T}'(\mathbf{z}) = \begin{pmatrix} -2x^2\mathrm{e}^{-x^2} + \mathrm{e}^{-x^2} & -2y^2\mathrm{e}^{-y^2} + \mathrm{e}^{-y^2} \\ 0 & -2y^2\mathrm{e}^{-y^2} + \mathrm{e}^{-y^2} \end{pmatrix},$$

therefore

$$\mathbf{J} = \mathbf{T}'(\bar{\mathbf{z}}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then by Theorem 3.4, any norm of **J** is strictly greater than 1.

Next we show that the equilibrium of the system generated by Eq. (5) is asymptotically stable. We can write this system as

$$x_{t+1} = x_t\mathrm{e}^{-x_t^2} + y_t\mathrm{e}^{-y_t^2} = x_t\mathrm{e}^{-x_t^2} + y_{t+1},$$
$$y_{t+1} = y_t\mathrm{e}^{-y_t^2}.$$

Let $f(x) = x\mathrm{e}^{-x^2}$. Since $f'(x) = \mathrm{e}^{-x^2} + x\mathrm{e}^{-x^2}(-2x) = \mathrm{e}^{-x^2}(1 - 2x^2)$, $f$ is strictly increasing in $[0, 1/\sqrt{2})$.

Define next $g(x) = x - f(x)$. Then $g'(x) = 1 - \mathrm{e}^{-x^2}(1 - 2x^2) > 1 - \mathrm{e}^{-x^2} > 0$ for $x \in (-\infty, \infty)$, that is, $g$ increases for $x \in (-\infty, \infty)$.

Select any initial state $(x_0, y_0)$. First we show that $y_t \to 0$ as $t \to \infty$. Since $|y_{t+1}| = |y_t| e^{-|y_t|^2} \leqslant |y_t|$, sequence $|y_t|$ is convergent. If $y^*$ denotes the limit, then the recursion implies that $y^* = y^* e^{-y^{*2}}$ showing that $y^* = 0$.

Select now an arbitrary $\epsilon \in (0, 1/\sqrt{2})$, and denote $\delta = g(\epsilon)$. Then $\delta > 0$. Since $y_t \to 0$, there is an $N$ such that $|y_t| < \min\{\epsilon, \delta\} = \delta$ as $t \geqslant N$.

Assume first that for some $t \geqslant N$, $|x_t| < \epsilon$. Then

$$|x_{t+1}| \leqslant |x_t| e^{-|x_t|^2} + |y_{t+1}| < f(\epsilon) + \delta = f(\epsilon) + g(\epsilon) = f(\epsilon) + \epsilon - f(\epsilon) = \epsilon. \tag{6}$$

Assume next that with some $t \geqslant N$, $|x_t| \geqslant \epsilon$. Then

$$\begin{aligned} |x_{t+1}| &\leqslant |x_t| e^{-|x_t|^2} + |y_{t+1}| < f(|x_t|) + \delta = f(|x_t|) + g(\epsilon) \\ &\leqslant f(|x_t|) + g(|x_t|) = |x_t|. \end{aligned}$$

Now we show that there is a $t^* \geqslant N$ such that $|x_{t^*}| < \epsilon$. Assume not, then $|x_t| \geqslant \epsilon$ for all $t \geqslant N$. Since sequence $|x_t|$ is decreasing when $t \geqslant N$, it converges to a limit $x^*$. Letting $t \to \infty$ in the recursion of $x_t$ and using the fact that $y_t \to 0$ we have

$$x^* = x^* e^{-x^{*2}} + 0$$

implying that $x^* = 0$ which contradicts the assumption. From the previous derivation we also see that for all $t \geqslant t^*$, $|x_t| < \epsilon$.

In summary, if $t \geqslant t^*$, then $|x_t| < \epsilon$, $|y_t| < \epsilon$, proving that both sequences converge to zero.

The above derivation also implies that the zero equilibrium is stable. Select any $\epsilon \in (0, 1/\sqrt{2})$, and define $\delta = g(\epsilon) > 0$. If $|x_0| < \delta$ and $|y_0| < \delta$, then it is easy to show that for all $k \geqslant 0$, $|x_t| < \epsilon$ and $|y_t| < \epsilon$. The monotonicity of sequence $|y_t|$ implies that $|y_t| < \delta < \epsilon$. Inequality $|x_t| < \epsilon$ can be proven by induction using inequality (6), since for $k = 0$ it holds, and the induction step is given by relation (6).

**Remark.** We can easily extend this example to any dimension $n \geqslant 2$ such that the equilibrium is asymptotically stable and the Jacobian at the equilibrium is a Jordan block with size $n \times n$. Just consider the dynamic system $\mathbf{x}_{t+1} = \mathbf{T}(\mathbf{x}_t)$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} x_1 e^{-x_1^2} + x_2 e^{-x_2^2} \\ x_2 e^{-x_2^2} + x_3 e^{-x_3^2} \\ \vdots \\ x_{n-1} e^{-x_{n-1}^2} + x_n e^{-x_n^2} \\ x_n e^{-x_n^2} \end{pmatrix}.$$

The unique equilibrium is $\bar{\mathbf{x}} = \mathbf{0}$. The Jacobian of $\mathbf{T}$ at this equilibrium is the $n \times n$ Jordan block with unit eigenvalue. Similarly to the discussion given in Example 3.6, we first show that component $x_n$ is stable and converges to zero. Then the same is shown for $x_{n-1}$, and then for $x_{n-2}$, and so on, and finally for $x_1$.

## References

[1] F. Szidarovszky, A.T. Bahill, Linear Systems Theory, CRC Press, Boca Raton, 1992.
[2] W. Li, F. Szidarovszky, Y. Kuang, Notes on the stability of dynamic economic systems, submitted.
[3] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, Cambridge, 1997, pp. 242–259.
[4] Y. Xu, Introductory Algebra (in Chinese), Shanghai Academic Press of Science and Technology, China, 1966, pp. 546–547.
[5] E. Kreyszig, Introductory Functional Analysis with Applications, Wiley, New York, 1978.
[6] K. Yosida, Functional Analysis, Springer, Berlin, 1980.
[7] J.M. Ortega, W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Serveral Variables, Academic Press, New York, 1970, pp. 42–44.